Possible gain of IT in problem oriented learning environments from the viewpoint of history of mathematics and modern learning theories

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Abstract: *In the first part we will outline that in history of mathematics eight activities proved to be fundamental for generating new mathematical knowledge. They can be taken as a framework for scaffolding mathematical learning environments in classrooms of today. By this, modern learning theories about constructivism as well as procedural and conceptual learning could be augmented and enriched.*

In the second part we will demonstrate by some mathematical examples for the middle and upper grades of high school the use of technology which might help to foster productive problem solving and thought processes. Furthermore ideas for a new computer based tool for measuring mathematical problem solving abilities in a PISA-like test are described which simulates some aspects of oral examinations. Finally we try to highlight in which way a computer-simulation of a mathematical lesson might help pre-service teachers to improve their abilities to teach mathematical problem solving.

1. Introduction

In this paper we want to highlight the utility of IT for the teaching and learning of mathematics in the middle grades with referring to some aspects which seem to be neglected until now.

First some elements of a theoretical framework are presented, which we draw from history of mathematics [37] and modern learning theories [10]. A framework should help to clarify educational goals and criteria for possible outcomes of learning-activities in mathematics.

In the following section we present a geometric problem-field which has its origin in the history of mathematics, too. A generalization of the Pythagorean theorem has been developed by al Sijzi (10^{th} century), which can be tackled already by 8^{th} graders and can lead up to elliptic curves, supported by DGS and/or CAS (cf. [2][1]).

2. Elements of a theoretical framework

2.1 History of mathematics

A long-term study of history of mathematics revealed eight main motives and activities, which proved to lead frequently to new mathematical results at different times and in different cultures for more than 5000 years (cf. [37]). We took this network of activities illustrated in Figure 2.1 as an element in our theoretical framework for the structuring of learning environments and for analyzing students' cognitive and affective variables.

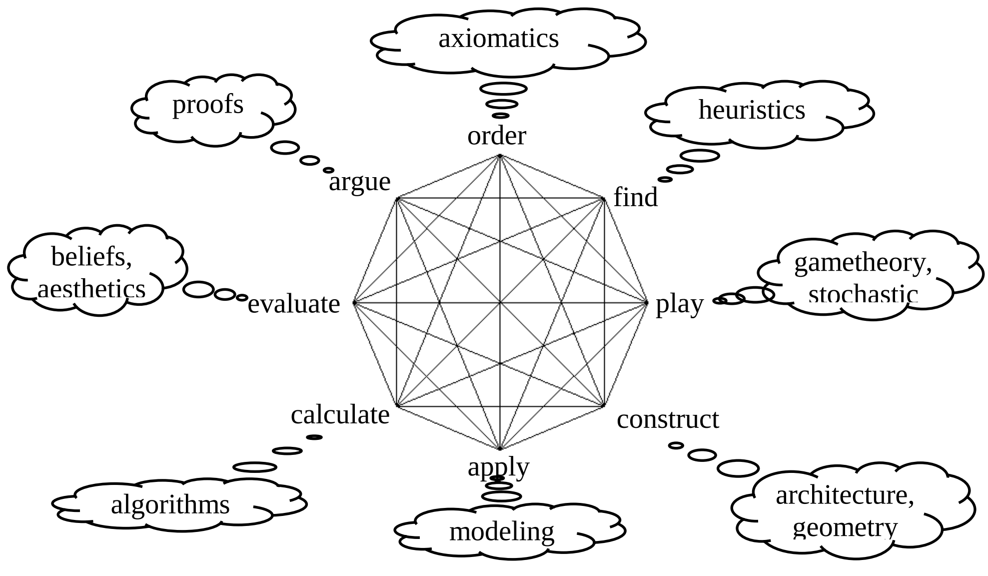


Figure 2.1 Network of activities which generated mathematical results along its history

Some explanation is given to Figure 2.1. There are three major groups of activities.

Calculating is at the beginning of nearly all mathematical actions. Problems, e. g., from astronomy and agriculture are until our days (cf. space industry and ecology) very important domains to ***apply*** mathematics or to develop new mathematical models, respectively. ***Constructing*** is the most important activity, not only in geometry but also in architecture – the latter one has been taken as a part of mathematics for a long time. These three activities are at the beginning of nearly all mathematical creations. We come now to a group of more sophisticated and challenging activities.

Arguing, esp. proving is at the core of modern mathematics and belongs to the more challenging mathematical actions. Of course, this activity is also related to ***finding*** methods (heuristics in the sense of Pólya), which lead to conjectures first. Without inventions there are no proofs! The tension to bring new knowledge, a set of new theorems or clusters of solved problems in a systematic ***order***, might lead in the upper grades to first approaches to axiomatization. This might help older and more mature students – practiced in appropriate situations and at appropriate time – to get a deeper understanding and more insight into theoretical connections.

The following two activities seem to be neglected rather often until now but proved to be of major importance for mathematical inventions, too. The striving for religious cognition and related systems of ***values*** generated frequently new problems and their solutions and produced in this way also new mathematical knowledge during history of mathematics. Systems of values are also frequently related to aesthetics, which may be sometimes still driving forces for mathematical inventions.

The same holds for an approach to mathematics by ***playing*** and the development of recreational mathematics. New branches of mathematics were very often created in this way like stochastic and game-theory.

These different activities - which are important, not only in mathematics - are connected and interrelated in many ways, which are represented in Figure 2.1 by "diagonals".

2.2 Modern learning theories

We want to highlight the importance of appropriate (modern) learning theories starting with two examples from the teaching-experience of the first author:

During his time as a high-school-teacher at a German Gymnasium he once taught the classical theme of solving systems of linear equations with two variables to his ninth graders. After coping with several concrete examples he wanted the students now to tackle and understand the general case of solving the system ax+by =c; dx+ey=f, considering the different possible conditions for solvability. One part of this enterprise was the development of an appropriate computer programme in the good old "BASIC"-language, to get a better feeling about the meaning and use of variables. At the end of a longer process of struggling with difficulties a pupil remarked: "Why all this abstract fuss? I only want to *understand* it, i. e., I want to have a simple method to get the right answer!"

Now a brand-new example: At the beginning of his last university-course "Introduction into Mathematics Education" the first author asked his student-teachers: Who could make already some experience with CAS during his or her school-time? Three students (out of 20, all female) said that they used CAS, more or less systematically, during their last three years in the upper grades of Gymnasium as well as in their final high-school-examination (Abitur).

The students were asked to report their experience with this specific knowledge in the background during their beginning mathematics courses at university (analysis and linear algebra). All three students said that they learned a lot about "pressing buttons" on their programmable calculators, but at the beginning of the courses at university they had – in relation to the other students without CAS-experience at school – a lack of theoretical understanding of the underlying concepts and relations of the prerequisite knowledge. Furthermore, it took them more time to adjust to a more appropriate learning attitude with focus on understanding.

The example from school can demonstrate that there are different ways to understand "understanding": *instrumental* (the expectation of the pupil) and *relational* (the expectation of the teacher). These terms were introduced by Mellin-Olsen and Skemp [31] and [34], respectively.

The example from the university-class can help to clarify, that there is *not only* this kind of polarity *in learning* but *also in teaching* of mathematics. In the same article Skemp additionally refers to "instrumental" and "relational" *mathematics*. Davis speaks in a similar context about "rote mathematics versus meaningful mathematics" ([3], p. 8) or routine vs. creative mathematics ([3], p. 14). But – and therefore the teacher was quite satisfied with the analysis of his students of their experience as pupils - they were now quite aware of the corresponding deficits which makes a good starting point to restructure their "instrumental bound" learning-schema. According to Skemp – but also to many other researchers – it is very difficult to re-arrange learning or teaching schema, which had been build up over a long period of time (cf. [32], p. 42, [34], p. 5, [35]).

A further possible consequence of the university-example is that it is not enough to have access and to use IT in the classroom. It should be done in a very reasonable and sensitive way. On this background some disappointing experience with using modern IT might be explained (cf. [26]).

Analogous polarities as between instrumental and relational learning can be found also in the terms *procedural* (cf. instrumental) and *conceptual* (cf. relational) knowledge, which according to Hiebert and Carpenter had been discussed already for many years ([15]).

These authors claim, that the most important question for future research is, not to precisely define these terms, but to ask for the relation between these two domains.
Especially this question had been analyzed recently very carefully by Haapasalo and Kadijevich [11].
We refer here especially to the work of Haapasalo et al. in [13].

In authentic actions performed by a person, procedural and conceptual knowledge can often be distinguished only by considering at which level of consciousness the person acts. Procedural knowledge usually involves automatic and unconscious steps, whereas conceptual typically requires conscious thinking. However, procedural knowledge may also be demonstrated in a reflective mode of thinking when, for example, the student skillfully combines two rules without knowing why they work. In order to be able to consider learning from a dynamic point of view, we adopted the knowledge type characterization of Haapasalo and Kadijevich [11]:

- *Procedural knowledge* denotes dynamic and successful use of specific rules, algorithms or procedures within relevant representational forms. This usually requires not only knowledge of the objects being used, but also knowledge of the format and syntax required for the representational system(s) expressing them.
- *Conceptual knowledge* denotes knowledge of particular networks and a skilful "drive" along them. The elements of these networks can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representational forms. Because the dominance of procedural over conceptual knowledge seems quite natural both in the development of scientific and of individual knowledge, it might also be pedagogically appropriate in mathematics to promote spontaneous procedural knowledge.

There are different possibilities to state logical relations between the two knowledge types. When we assume, that procedural knowledge is necessary for the conceptual, we are talking about the *developmental approach* or a *genetic view.* If it is assumed that procedural knowledge is necessary and sufficient for conceptual knowledge we speak about a *simultaneous activation view*[1]. Nevertheless, it seems appropriate to claim that the goal of any education should be to invest in conceptual knowledge from the very beginning. If so, the logical basis of this *educational approach* is the *dynamic interaction view* (i.e. conceptual knowledge is necessary for the procedural), or again the simultaneous activation view. This simultaneous activation view means that the learner has opportunities to activate simultaneously conceptual and procedural features of the current topic. By "activating" we mean certain mental or concrete manipulations of the representations of each

[1] Four views can be found in the literature on the logical relationship between conceptual and procedural knowledge, (cf. Haapasalo and Kadijevich [11]). The two approaches here are based on these views.

knowledge type. Being at the intersection of two complementary approaches, the simultaneous activation view is loaded with challenges concerning the planning of learning environments especially in the use of modern technology. Figure 2.2 illustrates the sophisticated interplay of the two approaches within the MODEM-framework as revealed in a large empirical project[2] (see Haapasalo [10]). It forms the basis for planning learning environments and for student assessment in a ClassPad project, even though the real learning activities are usually far from any systematic approach.

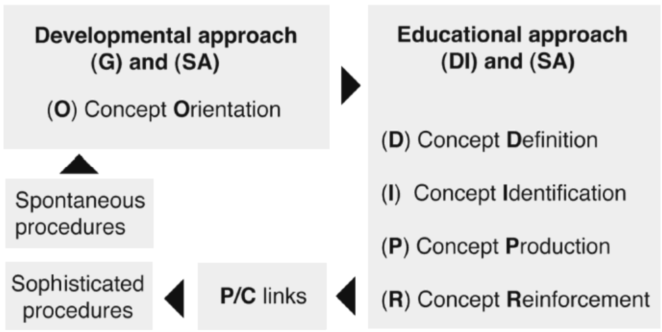


Figure 2.2 Simultaneous activation in the framework of the MODEM educational approach; cf. http://wanda.uef.fi/lenni/programs.html

We refer to this framework at the respective end of the following three sections.

3. Solving problems from history of mathematics and support of IT

The following two examples were taken from a collection of problems from al-Sijzī [29]. Abū Sa'īd Aḥmad ibn Muḥammad ibn 'Abd al-Jalīl al-Sijzī was a mathematician, who lived some 1000 years ago in Sijistan, which belongs today to Iran. Both examples were given for several times to different 9th-graders at a German Gymnasium by the first author and he reports his teaching experience. The task-formulation has been adjusted by him to modern times and to educational needs. The classes had some experience with similarity of triangles and the Pythagorean and related theorems.

3.1 A warming up problem

Problem 1:

Given a rectangular triangle ABC and its Thales-circle. Draw lines through A and B, which are tangent to the Thales-circle and perpendicular to *c*. The lines through CB and AC intersect these parallels in P and Q, respectively.

1. Investigate in which way the segments *r* and *s* (Figure 3.1) might be related to *c*.
2. Could you find more relations between the segments *r* and *s* and other segments in this figure?

[2] See http://www.joensuu.fi/lenni/modemeng.html

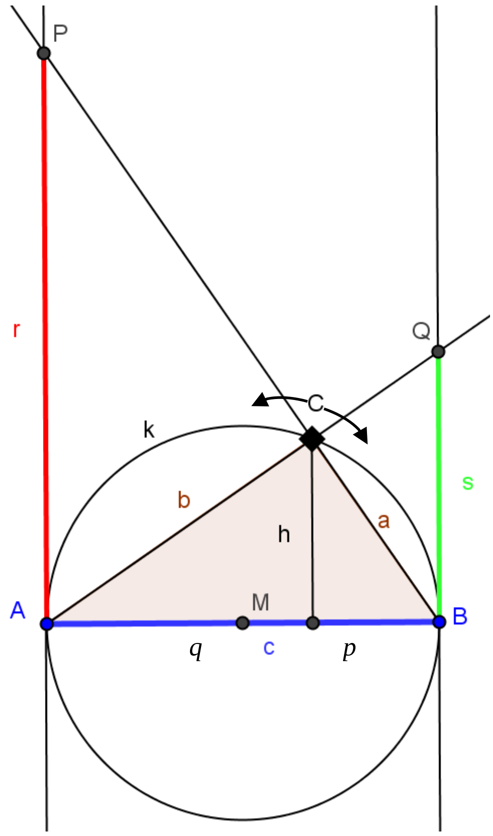


Figure 3.1

Solutions:
Part 1.: After some time the pupils found out by the help of DGS[3], that $r \cdot s$ is always a constant. They could find this conjecture by "moving" C on the circle. Changing the diameter of the circle triggered the assumption, that this product is related to the diameter.
Some pupils discovered without DGS, that the triangles ABC, BPA and QAB are similar. Therefore, e. g., the following equation holds:

$$\frac{r}{c} = \frac{c}{s} \tag{3.1}$$

So we get

$$r \cdot s = c^2 \tag{3.2}$$

I asked the students also for a geometrical interpretation of this equation. One of them said: "Let us reflect the triangle ABQ at the center M of c." In this way we came to Figure 3.2:

[3] All computer graphics (except Figure 3.9) were made with GeoGebra: http://www.geogebra.org/cms/en .

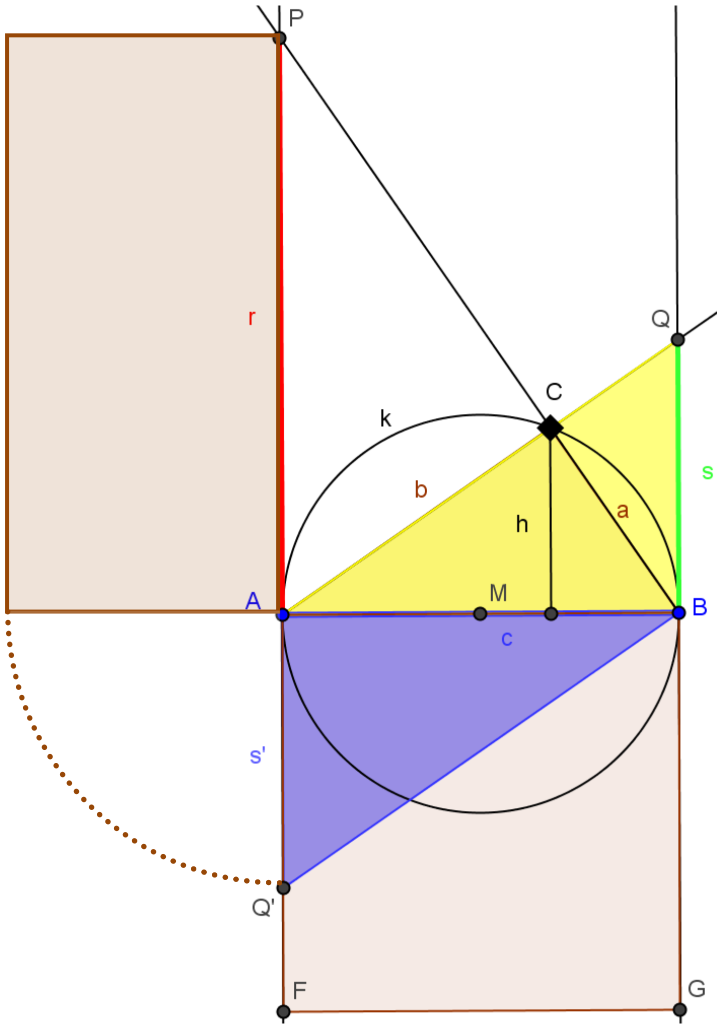


Figure 3.2 Geometric representation of c² = r*s,
cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziCircleParallelsnew.ggb

Putting the reflected triangle △(BAQ') and △(PAB) together, a larger rectangular triangle △(Q'BP) resulted with hypotenuse PQ' and rectangular angle in B. As c is its corresponding altitude and PQ' = $r + s'$, we can apply the well known theorem about altitudes in rectangular triangles

$$r \cdot s' = c^2 \tag{3.3}$$

which corresponds exactly to (3.2), as $s' = s$. So we could interpret equation (3.2) in a geometric manner, too: The area of the rectangle with sides r and s is as large as that of the square with side c. After starting with the first question the pupils carried out the following changes between geometric and algebraic representation which helped them to find a solution:

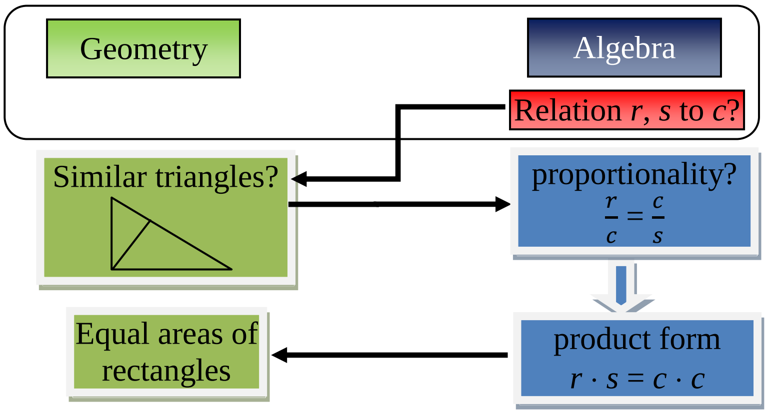


Figure 3.3 Problemsolving by change or representation

Part 2.: The experience from part 1 might help to set up a method to search for other relations:
- Looking for other equal proportionalities of sides in similar triangles or as a consequence of the intercept theorems,
- transforming these equations of proportions into equations of products,
- trying to interpret these equations of products as equations between the areas of rectangles.

Let us take an example: $r : h = c : p \Leftrightarrow r \cdot p = c \cdot h$. The right term can be interpreted at once as the area of a rectangle with length c and breadth h. As r and p are not directly connected, there is no direct interpretation of $r \cdot p$. If we substitute p by $c - q$, we can create a more comfortable situation:
$r \cdot (c - q) = c \cdot h$; so $r \cdot c - r \cdot q = c \cdot h$ or $r \cdot c = c \cdot h + r \cdot q$.

There might be found similar relations by students, practicing problem solving and finding by changing between algebraic and geometric representation as presented in Figure 3.3.

3.2 A generalization of the Pythagorean theorem

Problem 2:

Given a rectangular triangle with its Thales-circle. “Move” A and B in such a way out of the Thales-circle, that the new points A’ and B’ are also located symmetrical to the center M of this circle. “Move” now C on the old Thales-circle.

What can you figure out about the sum of the squares $(a')^2+(b')^2$? Compare with $(c')^2$!

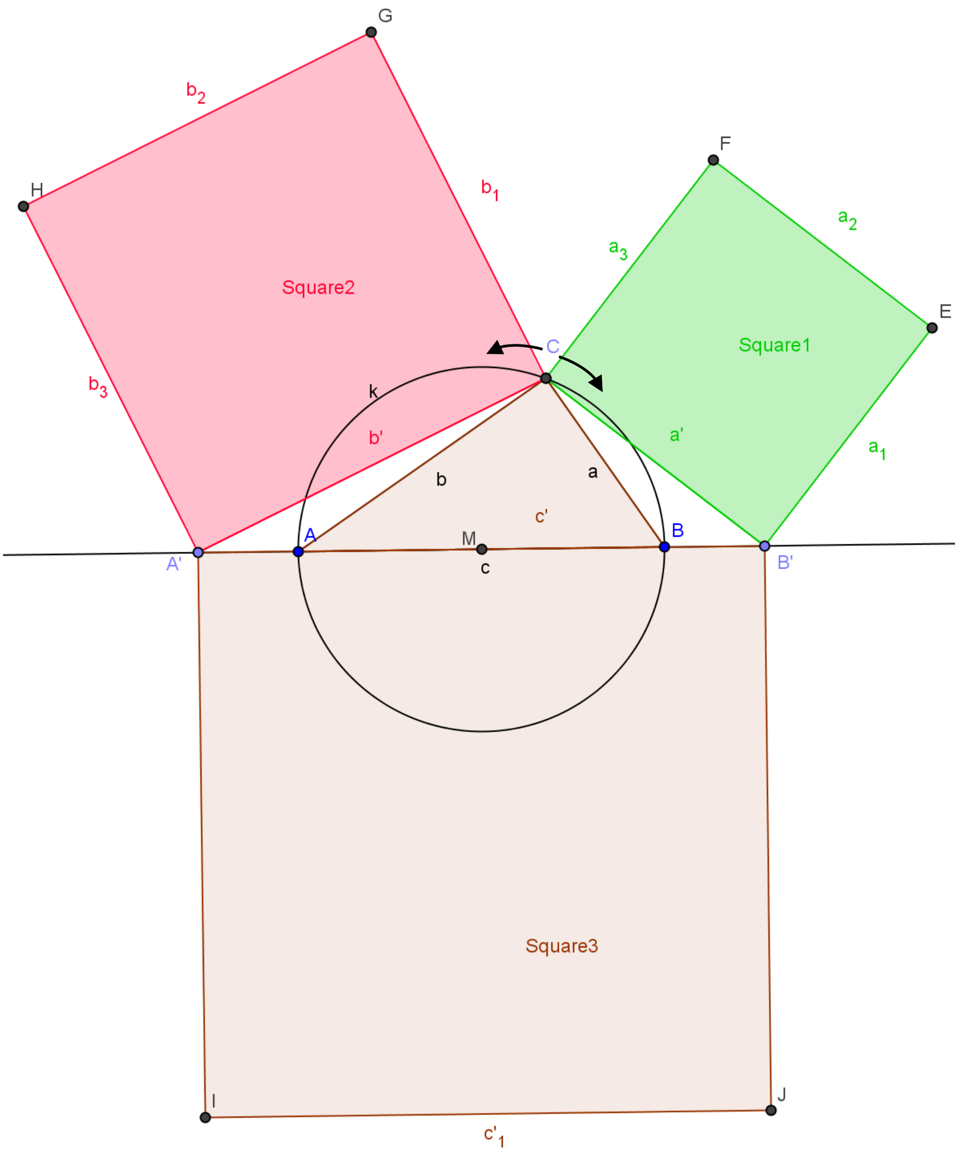


Figure 3.4 Comparing the squares when moving C along the circle;
cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziPythagorasGeneralized0.ggb

A draft from the original version of this problem is given in Figure 3.5:

Figure 3.5 Part of the manuscript of al Sijzi (copy from microfilm)

Solutions:

Pupils come rather often to the conjecture $(a')^2+(b')^2$ = constant spontaneously. In case of no ideas again DGS can help.

A proof can be carried out in several ways (and so has been done by al-Sijzī [29]). One possibility is given here. By examining the special degenerated case (cf. Figure 3.4) by dragging the point C to the point B, the conjecture can be posed more precisely in the following way:

Theorem 3.1: $$(a')^2 + (b')^2 = (c + d)^2 + d^2 = \text{const.} \tag{3.4}$$

This statement can be represented by the following Figure 3.6 ($(c + d)^2 + d^2 = c^2 + 2cd + 2d^2$):

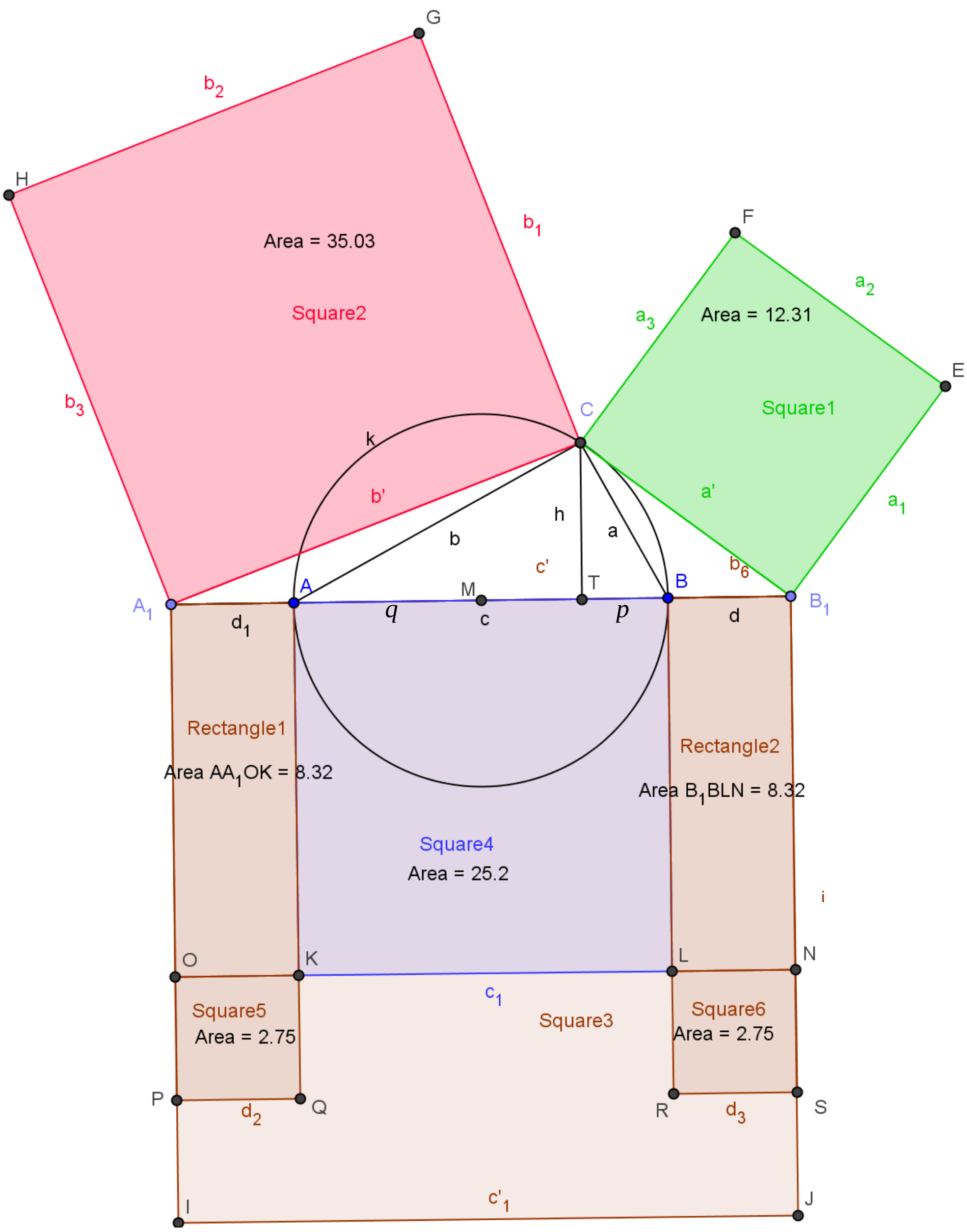


Figure 3.6 Representing the area of square 1 and square 2 by parts of square 3; cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziPythagorasGeneralized1.ggb

Proof: Applying two times the Pythagorean theorem, we get the following two equations:

$$\begin{aligned}(a')^2 &= h^2 + (p + d)^2 = h^2 + p^2 + 2pd + d^2 \\ (b')^2 &= h^2 + (q + d)^2 = h^2 + q^2 + 2qd + d^2\end{aligned} \tag{3.5}$$

we add these equations and receive

$$(a')^2 + (b')^2 = 2h^2 + p^2 + q^2 + 2dc + 2d^2 \tag{3.6}$$

we substitute h^2 by $p \cdot q$, using a well-known theorem and the fact $c = p + q$, and come to:

$$(a')^2 + (b')^2 = 2(p \cdot q) + p^2 + q^2 + 2dc + 2d^2 \tag{3.7}$$

so we get

$$(a')^2 + (b')^2 = (p + q)^2 + 2dc + 2d^2 \tag{3.8}$$

and finally

$$(a')^2 + (b')^2 = c^2 + 2dc + 2d^2 = \text{const.} \qquad \blacksquare$$

Problem 3: Generalization of Problem 2

Al-Sijzī presents a first simple generalization: He “moves” the points A, B *into the inner* of the Thales-circle. He proves, that $(a')^2 + (b')^2$ =const. holds also in this case. We get

$$(a')^2 + (b')^2 = (c - d)^2 + d^2 = \text{const.} \tag{3.9}$$

As (3.9) can be written also in the form

$$(a')^2 + (b')^2 = c(c - 2d) + 2d^2 = c \cdot c' + 2d^2 \tag{3.10}$$

the expressions in (3.10) can be interpreted as areas of corresponding polygons in Figure 3.7, too.

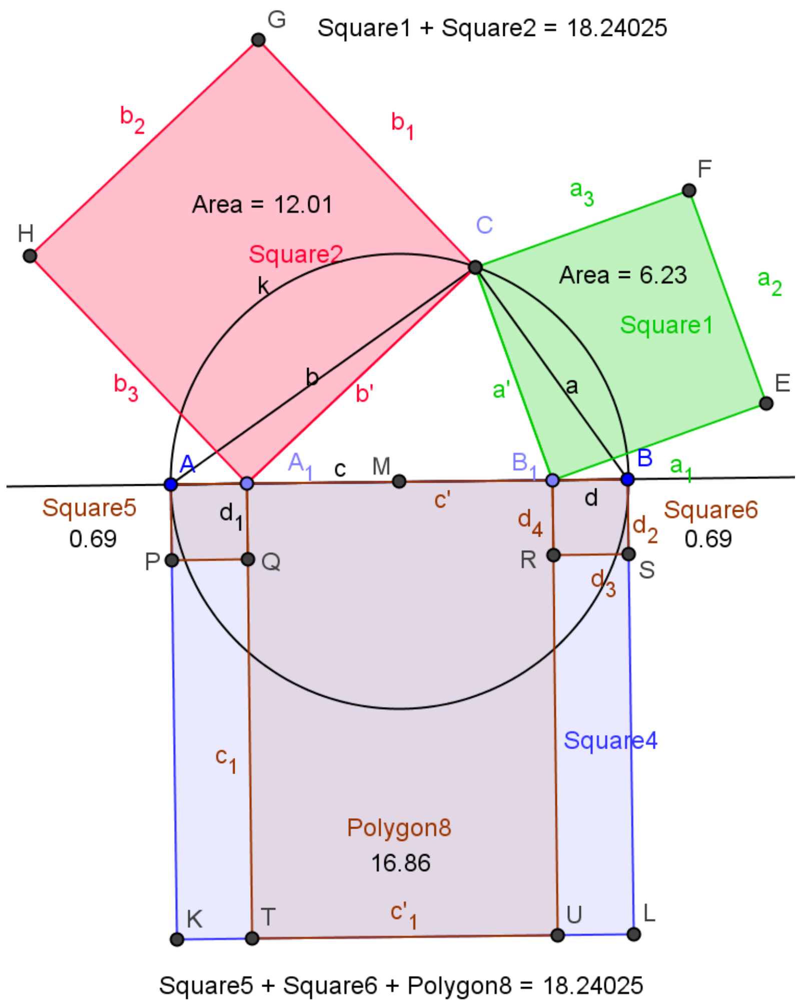


Figure 3.7 Moving A and B into the Thales circle and representing square1 and square2 in square4; cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/alSijziPythagorasGeneralizedinternal3.ggb

3.3 Conversion of problem 3 and generalization

Problem 4:

Now we consider the conversion of problem 3:

Given two fixed points A and B in the plane. What is the locus of all points C in the plane with the property AC^2+BC^2=const.?

Solution:

It is quite clear to assume that this locus has to be a circle. We refer to Figure 3.7 and redefine A=A_1, B=B_1, AB=c, AC=b and BC=a. Choosing A as the origin of a coordinate-system, the starting situation is represented in Figure 3.8:

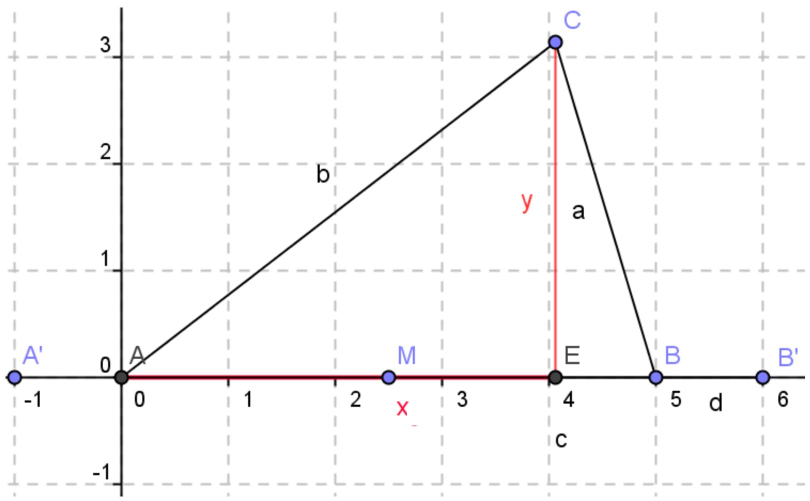


Figure 3.8 Transforming equations for drawing graphs in a coordinate-system

By using the Pythagorean theorem we get the two equations

$$a^2 = y^2 + (c - x)^2$$
$$b^2 = y^2 + x^2 \tag{3.11}$$

By addition and some simple transformations - using (3.10) - we get

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2} + d\right)^2 \tag{3.12}$$

which is the equation of a circle.

Problem 5: Generalization of problem 4

Given two fixed points A and B in the plane. What is the locus of all points C in the plane with the property

$$AC^n + BC^n = \text{const.} \Leftrightarrow a^n + b^n = (c + d)^n + d^n),\ n \in \mathbb{N}? \tag{3.13}$$

Solutions:

It is obvious, that in case $n = 1$ we get an ellipse. In case $n > 2$, using the notation in Figure 3.8 and equations (3.11), we get

$$a = \sqrt[2]{y^2 + (c - x)^2}$$
$$b = \sqrt[2]{y^2 + x^2} \tag{3.14}$$

so we have to solve the equation(s)

$$\sqrt[2]{y^2 + (c - x)^2}^{\,n} + \sqrt[2]{y^2 + x^2}^{\,n} = (c + d)^n + d^n \tag{3.15}$$

For c=5, d=1 and n = 4 resp. n = 8 MathCad suggests solutions, which can be represented as follows:

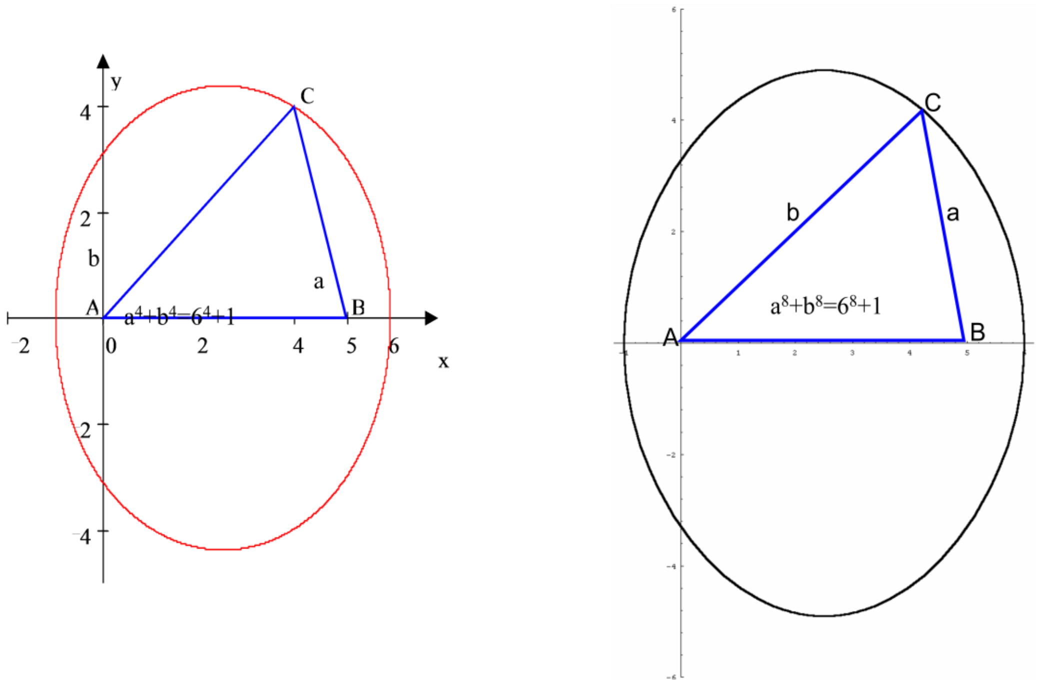


Figure 3.9 Fermat-curves of degree 4 resp. 8;
cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/FerSijziMC.htm

Here we have two examples for Fermat-curves (cf. e.g. Schmidt [30]). Further exploration can be carried out in the upper grades (exploring relations to the famous Fermat-theorem!).

3.4 Reference to our theoretical framework

- A lot of activities are supported when investigating these examples which proved to be important in history of mathematics: find, order and prove seem to be quite necessary when pondering in this problem field. Of course some simple calculations are to be done as well, but might be delegated to computer-software mainly.
- Simultaneous activation of procedural and conceptual knowledge is strongly involved when coping with the presented problem field. As already mentioned simple calculations are necessary but it should be intensively interwoven with conceptual (relational) methods, especially when to decide whether computer should be applied or not (depending, of course, very much on prerequisite knowledge).

4. IT as an aid to better assess mathematical thinking- and understanding processes in PISA-like tests

During the last 15 years (as a consequence of TIMSS and PISA) a distinct shift happened from input to output-orientation in mathematics education. There had been a long discussion about the effectiveness of assessment-instruments as standardized tests (cf. Zimmermann [36], esp. Hilton resp. Lax & Groat in Steen [33], p. 79, resp. 85).
In spite of the fact that there are, meanwhile, quite useful standards for assessment (cf., e. g., [24]) and developers of tests like PISA don't use the multiple-choice format only, there is still a lot of criticism (cf. e.g. Jahnke & Meyerhöfer [19]).

One type of criticism refers to the content-validity of the problems (cf. Kießwetter [20]), another one to the test-format (cf. [19]). Even in case that there is the possibility for a “free” response to an item, a false response does not mean, that the subject is not able to do mathematics. Often one cannot exclude that the subject did not understand the intention of the test-developer (cf. e.g. Wuttke in [19], p. 144). Furthermore, it is often rather difficult to make a correct interpretation from the solution- remarks of a pupil. Additionally, the qualification of the test-evaluators is not always very high – as it might happen rather often in a large test-enterprise as PISA.

In oral examinations it is much easier for a competent examiner than in conventional tests to get a better understanding of the mathematical understanding of a candidate. E. g., after the pupil has given a wrong answer, the interrogator can pose a simpler question (referring to a special case). In case of a right answer, the examiner can check the understanding (thus reducing random-effects) by posing similar or more general questions.

On this background the idea was born to develop a computer-aided test-design, which should simulate some aspects of an oral examination (cf. Rehlich [27]).

Besides the goal of getting a deeper understanding of mathematical competences of a pupil a computer-program might have the advantage, that it is more reliable than normal (different!) evaluators and could help - by tracing the results of the inputs – to be less subjective (and save a lot of man-power!).

4.1 The problem environment and simulation of an oral examination

Our starting point was the following well-known PISA-problem:

> A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. Which pizza is better value for money? Show your reasoning.

It is obvious that one can get a more sound estimation of the potential of achievement of the tested subjects, if one tries to take into account parts of their different networks of knowledge and their different habits of communication. How might this be realized? In an ideal situation one should have the possibility to learn much about the subject over a long range of time, e.g., by observations or/and by long interviews about specific contents. But, of course, because of economical and organizational reasons already the latter procedure cannot be carried out especially in case of a large group of subjects to be tested. Furthermore, if different interviewers would be necessary, their procedure – especially their reactions - cannot be standardized. Think only of different nonverbal reactions like gestures and facial play! So in addition of a lack with respect to economy the outcomes of different interviews can hardly be compared.

But by a branching computer program one might try to reconstruct this type of methodological interventions, which are possible in an interview and which make this more informative than a normal written examination:

- you can reformulate questions,
- you can present hints with a different range of influence,
- you can ask, what kind of solution method is preferred by the subject,
- you can present analogous situations to be analyzed by the subject,
- you can test the subjects’ ability to generalize with respect to a specific insight.

You can program the computer in such a way, that the working process of the subject can be traced and saved by the computer. In this way it might be possible to get a more precise reconstruction of the „hidden“ thought-processes than by a mere comprehensive paper-and-pencil test with many items, which allows very often only the judgment of right or wrong. Even in case of more comprehensive written solutions there is very often a severe problem of interpretation.

To outline our intention, we try to clarify this by a possible modification of the foregoing pizza-example. We start in the program with a variant of this pizza-problem:

> The Miller family bakes some cookies for a Christmas-fair. To cut out the cookies they use two circular forms with a diameter of 4 cm and 6 cm, respectively. They want to produce small packed portions including four large cookies or a corresponding amount of small cookies. How many small cookies should be taken to get a bag of the same weight as a bag with four large cookies?

The subject works on this entrance-task and can choose one of several possible answers. The reaction of the computer depends on this answer. In case of a correct answer the program presents to the subject a follow-up question:

A follow up question: How many cookies you should take,
if they have a diameter of 2 cm only?

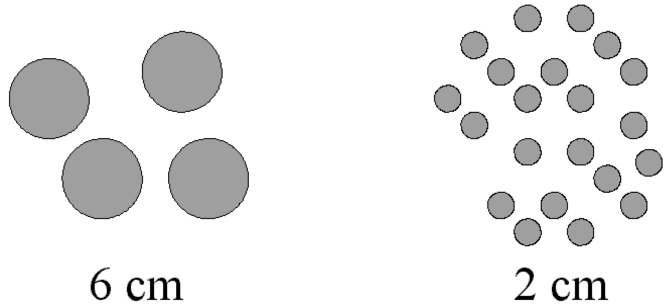
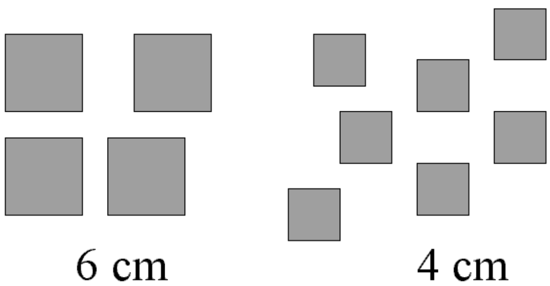


Figure 4.1 A follow up question

During the ongoing process – as far as the subject makes the right input – it is checked by an appropriate sequence of problems, to what extent the subject is able to use generalized concepts or – at least – is able to develop them quickly. In case of a wrong answer there are different reactions, corresponding to the specific type of mistake. E. g., if the subject responds “6 small cookies”, this might be interpreted as a strong hint that there is a wrong use of the concept of proportionality (between diameter and area) which might have proved to be very successful to the subject in many other cases. Therefore we have a reaction of the computer by presenting an analogous problem with quadratic cookies:

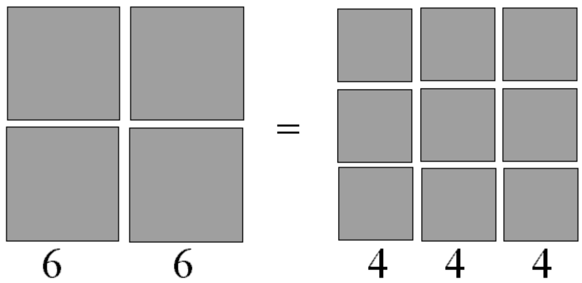


How many small cookies have the same weight as four large cookies?

Figure 4.2 Going back to squares

This variation might yield to the subject some reduction of the complexity of the initial problem. The information, how the subject tackles this problem, might be very useful to esteem his achievement potential. After successful working on this problem the following question is given to the pupil:

> Now we would like to know which of the following solutions, presented here, is most similar to your own way of solving the problem.
>
> Method 1: calculation:
>
> 4 quadratic cookies with the side-length of 6 cm each and an area of $A = 4 \cdot 6^2 \text{ cm}^2 = 144 \text{ cm}^2$.
>
> 1 cookie with side-length 4 cm has an area of $A = 16 \text{ cm}^2$.
>
> Therefore 144:16 = 9 small cookies are necessary.
>
> Method 2: geometrical considerations:



9 small squares cover as much area as 4 large squares.

Figure 4.3 Detecting different solutions

In order to get information about the subject's procedure it is very useful to esteem which type of follow-up question might fit to the thinking-tools used by the subject.

In the further course of the program the subject – who failed until now - is led back to circular cookies. Dealing with the simpler problem might help the subject to unlock a state of blockade or might help to make him conscious that his first reaction to the starting problem was not appropriate.

4.2 The structure of the computer guided test

These few examples should help to clarify that the structure of this computer-guided test is established by a network of different possible sequences of actions of the subject.

It is obvious, that it makes a major difference whether the pupil knows already a solution method for this type of task/problem or whether the given problem is completely new to him. In the first case he has only to solve a task by activating "knowledge and skills", in the second case he has to solve a real problem. So heuristics are in demand. In case of such creative activity pressure of time of a testing situation has much more influence than in case of solving a routine task. In this case very often quite different strategies of exploration are used by different pupils. Before constructing such test one might get some overview what type of activities might be expected generally by goal-oriented observation of pupils. The following picture represents some possible paths in the computer program which might be followed by a pupil.

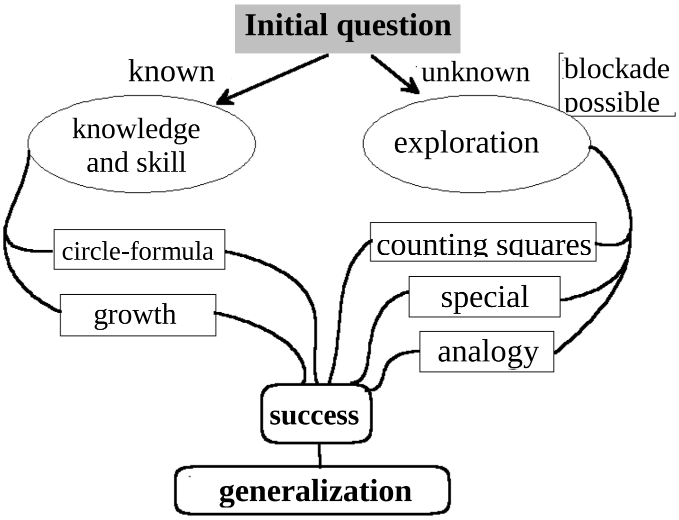
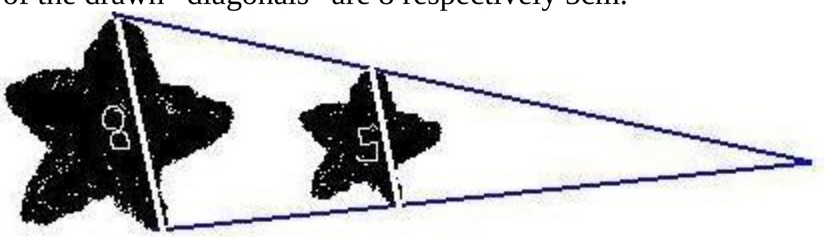


Figure 4.4 Main paths in the computer program

In case of activation of "knowledge and skill", it might happen, e. g., that the pupil calculates areas by using the rule of three. This can be carried out more or less clever. But perhaps the subject knows also some abstract concepts of growth (area is growing by the second degree …). Another point of difficulty may be established by the fact that the ratios of the diameters of the cookies are no integer numbers. As to my experience the transition from integer ratios to ratios which are fractions is a great intellectual jump in quality which asks for more potential of abstraction.

Both routine paths sketched above might lead to success, but they can lead also to a dead end – in case of mistakes in calculations or flaws in thinking. Depending on the individual way to success the computer can present different follow-up problems. When the pupil applied the formula for the area of a circle it can be checked, in which way he might react when shapes are presented to him for which he does not know any formula or ready-made strategy:

> Now we take cookies which look like ("perfect, mathematical") stars. The cookies of the larger type are in true scale to the smaller ones.
> The length of the drawn "diagonals" are 8 respectively 5cm.



> What is the weight of 100 large cookies, if 100 small cookies have a weight of 2000g?

Figure 4.5 Exploring the understanding of the concept of similarity

Here is an example of a page of a first version of the program that had been developed by Rehlich in Visual Basic:

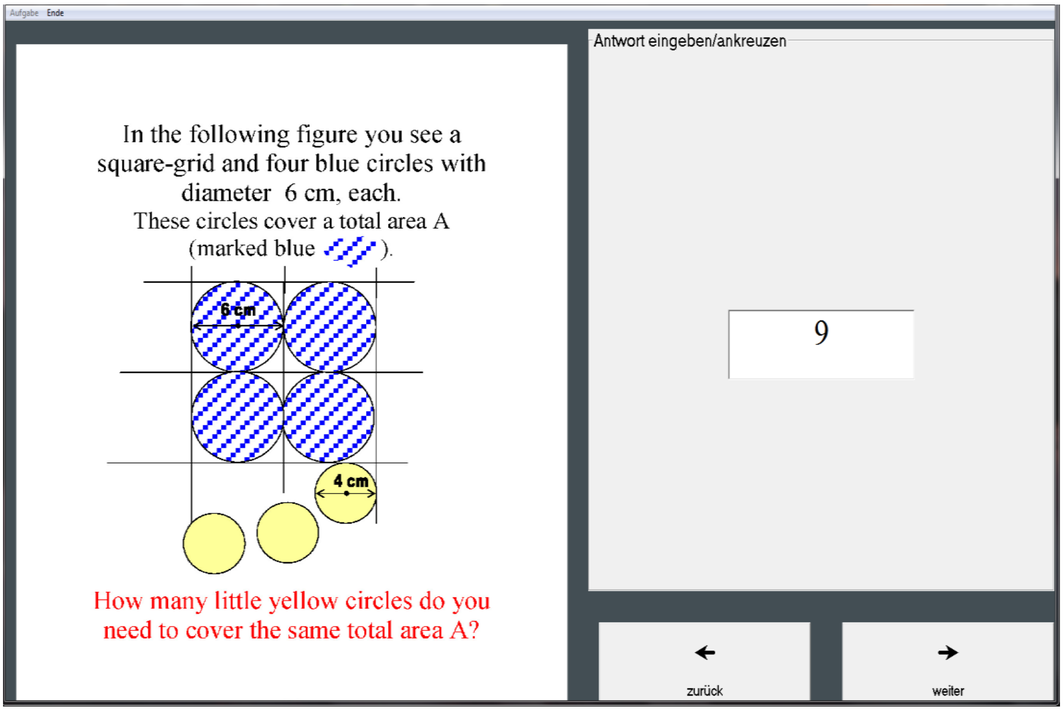


Figure 4.6 Hardcopy of a program-page;
cf. http://www.remath.de/CTest/Start.htm

4.3 Additional remarks to the testing instrument

It takes a lot of energy and time to produce a test like we have described. A major part of this expense is to be given to a detailed analysis of the appropriateness of the problems and tasks to be used. Only after sound and intensive experience with pupils, thinking and reacting in different ways, one can get a feeling for the complex interaction and interrelation between presentation of the problem and pupils' approaches to the problem.

On the other side, the high expense corresponds to a high effect: The testing tool reacts less sensitive to the impact of different habits of talking and expectation with respect to the outcomes of the tests than a written examination, which might be misunderstood by the pupil. This is due to the implemented possibility to press the pupil on the initial question by additional questions.

Furthermore, more detailed information can be gained about possible reasons for good or bad results. In this way one can get a more comprehensive and expressive picture about strengths and weaknesses of pupils or groups of pupils. Such information can also be useful for the evaluation of lessons already held and for the planning of future lessons.

4.4 Reference to our theoretical framework

- At least abilities to find, argue, order and calculate can obviously be tested by this program.
- It seems to be clear that it should be possible to come to a more comprehensive survey of pupils' conceptual understanding and its relations to his procedural knowledge.

5. How to teach problem solving – some help by computer-simulation

5.1 Motivation and goal of the study

Problem solving had been on focus in mathematics education already for decades – not only since the NCTM published its "Agenda for Action" in 1980. Nevertheless, all over the world the implementation of effective teaching of problem solving into the classroom is still on a broad range a major difficulty (cf. e.g. Thompson [35]). One reason is the fact that teaching mathematical problem solving is a very difficult and itself a very complex problem.

Solving complex problems has been a domain of cognitive psychology for more than three decades (cf. e.g. [4], [5]). In this context programs "simulating" e.g. developing countries and little cities had been developed and problem solving behavior of subjects "governing" these institutions (by using these programs) had been analyzed ([4]).

It had been the idea of Kießwetter, to transfer this approach to mathematics instruction (cf. [20]). This idea had been realized by Fritzlar [6], [7]. We present now a short overview about this work; skipping here the theoretical background (which is carried out comprehensively in [7]).

A computer-program was developed, which simulates several important aspects of the instruction of mathematical problem solving. This was given to student teachers and their behavior was documented when coping with this program. The *main goal* was to determine to what extent these students were already *sensitive for the complexity of problem oriented mathematics instruction* ("POMI"). Therefore a more detailed *operationalization for "sensitivity for complexity"* had to be carried out and to be tested.

Later on, the program might be used as an *additional* tool *to improve teacher students' sensitivity* for such complexity.

5.2 Procedure

A main element of the reported study is the following problem:[4]

> The "paper-folding-problem" (*Faltproblem*): A sheet of usual rectangular typing paper (A4) is halved by folding it parallel to the shorter edge. The resulting double sheet can be halved again by folding parallel to the shorter edge and so on.
> After n foldings the corners of the resulting stack of paper sheets are cut off. By unfolding the paper, it will be detectable that (for $n > 1$) a mat with holes has resulted. Find out and explain a connection between the number n of foldings and the number $A(n)$ of holes in the paper.[5]

[4] This problem was developed by Kießwetter to be used in an entrance examination of the University of Hamburg.
[5] Used formulation and goal of examination are intended for the teacher. There are many possibilities to communicate this problem to pupils. Different ways of posing the problem were incorporated into the program.

Within the study teachers, students, and the author of this study ([6]) tried out this problem in about 50 lessons mainly in fourth and fifth grades (age 10 to 11) of different school types. These experiences showed special potentials of the *Faltproblem* for POMI, which can only be listed here:[6]

- The problem can be understood very easily (also by young pupils), nevertheless it is not at all mathematical simple.
- Many possibilities to attempt to come to terms allow a differentiated work on the problem.
- Very often pupils can make conjectures, search for explanations and, more generally, work heuristically.
- The problem is a motivating challenge for pupils. Generally they enjoy working on it.
- There are many possibilities for communication and cooperation.
- The problem is open with respect to different ways and also to different goals of working on it.
- There are many relations to several mathematical subject areas and other mathematical problems (e.g. “Tower of Hanoi”).
- Many variations and extensions are possible.

Pupils’ independent work on the *Faltproblem* brings additional demands on the teacher with regard to mathematics, but also (not independent from these) to realization of the lesson. Relating to the second area, we see special demands concerning

a) planning suitable lesson scenarios and corresponding teacher actions to initiate and maintain productive activities of pupils, and
b) analyzing of present lesson situations appropriately with attention to relevant aspects and with a suitable extent to go into detail.

Two genuine situations should do to intimate these demands on the teacher:

Situation A from a fourth grade of a primary school:

As a starting point to work on the problem, teacher and pupils fold a sheet of paper twice. Before it will be opened the teacher asks the pupils to make conjectures. Here are some:

`*There will be two holes in the paper, where it is not open.*`[7]
`*There will be only one hole and triangles at the edges.*`
`*There will be five holes.*`

After unfolding the paper sheet the pupils quickly agree that there is only one hole in the paper. The teacher writes down the result into a table on the blackboard when Maria suddenly argues:

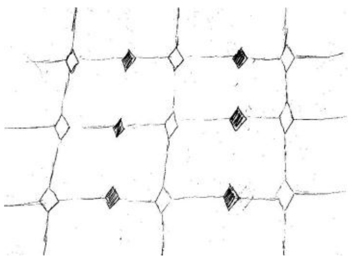
`*You could also say that there are three holes in the paper, because there is a half hole at every edge.*`

- How should the teacher react to this situation?
- Should he bring Maria’s comment up for discussion?

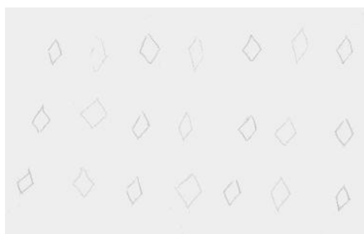
[6] For more details see Fritzlar [7].
[7] Original statements of students, slightly modified for better understanding, are marked by asterisks (*).

- Should he reject it, because “half holes” do not exist? And after all almost all students agree that there is one hole in the paper. But isn’t Maria right by comparing the shape of holes and notches? And how would she react to this rejection?
- So should the teacher accept Maria’s remark? But didn’t he just agree to the statement that there is only one hole? Is it not right any longer?
- Or should the teacher allow both answers? But, in this case, how to cope with the cut off - corners shaped like a “quarter of a hole”? …
- Should he start a discussion about a definition of a “hole”?
- Are the pupils used to work on a problem in several ways?
- Which arithmetical patterns can be found and how useful can they be?
- Is the idea of “half holes” helpful for solving the problem? …

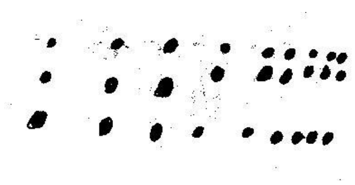
Situation B from a fourth grade of primary school:
First, teacher and pupils had folded and cut the paper sheet several times. Then the pupils had been instructed to sketch, how the paper sheet will look after the fifth folding-cutting operation (before unfolding it). They were also supposed to distinguish between “old” and “new” holes. After all, some pupils presented their results:



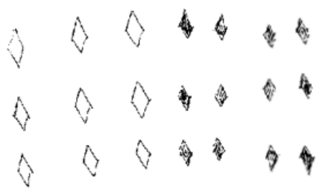
Claudia: `*I think holes result from folding lines.*`



Johannes: `*Between old holes are new ones. So there are 21 holes.*`



Dagmar: `*I think there are simply 27 holes.*`



Sibylle: `*It has doubled.*`

How are these results to be interpreted? At a first look, only Johannes fulfilled the job correctly, but Claudia achieved an important partial result too - she constructed a connection between holes and folding lines. Dagmar apparently ignored geometrical aspects of the paper sheet. This might imply an arithmetic viewpoint which also explains her result: The number of holes triples every folding. Sibylle presumes 21 holes too, but she comes to another arrangement of holes. Can it be justified?

Fritzlar designated (in the sense of a provisional working definition) a person as sensitive for the complexity of math-cognitive aspects of POMI, if he is aware of the complexity of POMI, of special demands arising from it and of limits of his possibilities to decide and to act in an appropriate way in mathematics instruction. It seems to be clear that sensitivity appears above all in investigation and evaluation of decision-situations connected to POMI (as both ones described above). That's why Fritzlar developed a realistic and interactive computer scenario – based on interviews of students, teachers and teacher educators and on almost 50 lessons with the *Faltproblem* –, which confronts the user with such decision-situations.

In this scenario the *Faltproblem* can be virtually taught in three different classes. For that the user takes the part of the teacher.

- At first he can choose a *class* (out of three) he wants to teach.
- Then he can decide about *goals* of the lesson
- He decides *how to begin* the work on the problem.
- Then the scenario models some possible and probable *reactions of the pupils*, especially their working processes and (partial) results, taking especially the 50 lessons, already mentioned, as possible resource (notes of real reactions of pupils, video spots, pictures)
- The *user has to react* again.
- But he can *also go back* and correct his former decisions or give some additional alternatives for reaction.[8]
- At the end of the lesson an *assessment of his decisions* particularly with regard to his lesson's goals is presented to the user.
- From this point he can also go back to former decision-situations or start all over again.

5.3 Results

Result 1: The program

Here we have the homepage of the program[9]:

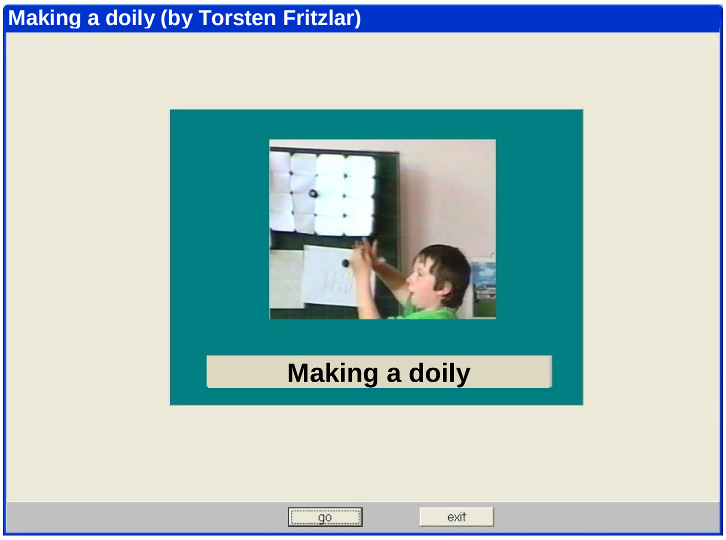


Figure 5.1 Homepage of the program

cf. http://www.mathematik.uni-jena.de/~bezi/Vortraege/TorstenKomplexiProgr10112011short.pptx

[8] The scenario cannot react on alternatives given by users. But the user could write them down and they were automatially collected and can be used for further development of the scenario.

[9] Programmed by Fritzlar in DELPHI.

The following diagram illustrates some possibilities of interaction with the computer scenario:

choice of a class
statistical data, pupils' abilities, experiences concerning
working on problems, collaborative working, ...

choice of goals of the lesson / ranking of the goals
short-term goals (regarding the problem working on),
long-term goals (regarding heuristic competences, attitudes, ...)

decisions during the lesson
way of presenting the problem, questions and instructions to the pupils,
selecting material and media, ...
Given alternatives can be chosen or completed.

reactions regarding lesson situations, pupils' problem solving processes and results
pupils' activities, conjectures, argumentations, (approximate) assessment
concerning pupils' motivation, involvement, …

assessment of the lesson
comparison of achieved and planned goals, assessments concerning
consistency of decisions, extent of control by the teacher,
comprehensiveness of mathematical activities, ...

Figure 5.2 Main path and possibilities of interaction with the computer scenario

Result 2: Operationalization of "sensitivity for complexity" (SFC)

The following „SFC-vector " with 4 components was created on the basis of preliminary theoretical considerations as well as on the basis of experience with the use of the computer scenario. It should help to represent more precisely the degree of SFC of teacher students, indicated by several data when using the simulation-program:

1. exploratory behavior
2. context sensitivity
3. inconsistence
4. awareness, reflectivity

ad 1. Exploratory behavior

This component should represent the quantitative aspect (# of loops and of jumps back within the program: "amount") as well as qualitative aspects of the exploration of the scenario (# of different modes of representation of the problem: "range").

Graphic representation:

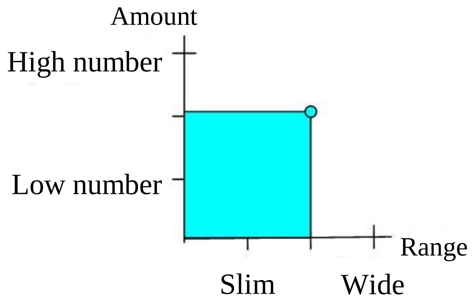


Figure 5.3 Exploratory behavior

ad 2. Context sensitivity

This component should help to represent to what extent the student-teacher referred in decision-situations to problem solving processes of the pupil (pink), aspects of the mathematical content (yellow), or more social aspects (motivation, teaching methods; green; in previous or following situations) of the "lesson" (program).

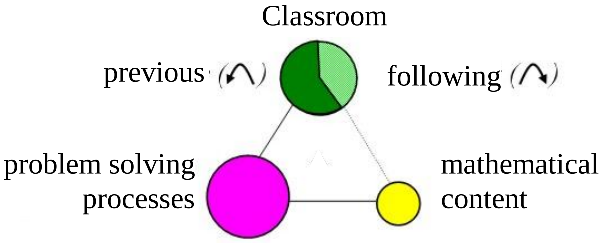


Figure 5.4 Context sensitivity

ad 3. Inconsistence

This component should represent the amount of decisions which were interpreted to be *not* consistent with educational goals selected at the beginning of the session.

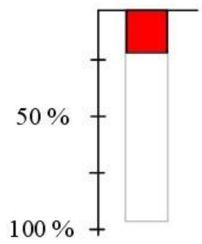


Figure 5.5 Inconsistence

ad 4. Awareness, reflectivity, metacognition

This component should represent on a rating scale the degree of (critical) reflectivity with respect to own decisions, possible connections about different aspects and thoughts about additional alternatives (this scale is related to the relative differences between the subjects in this respect).

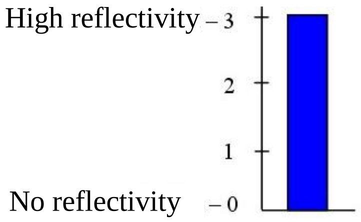


Figure 5.6 Awareness, reflectivity

Result 3: Students' SFC-profiles

We select two student-teachers, who represent two extreme cases: one with the highest scores concerning the SFC-vector and one with the lowest scores:

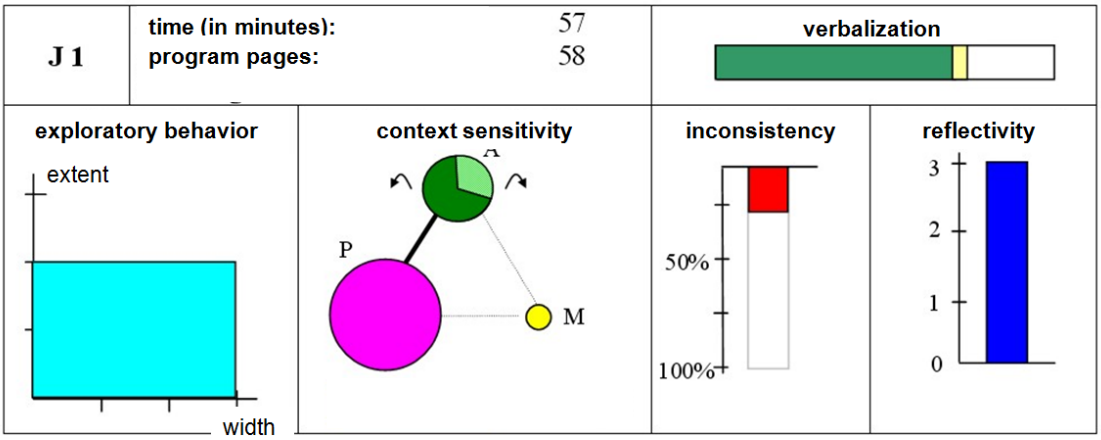


Figure 5.7 *High SFC* on nearly all components

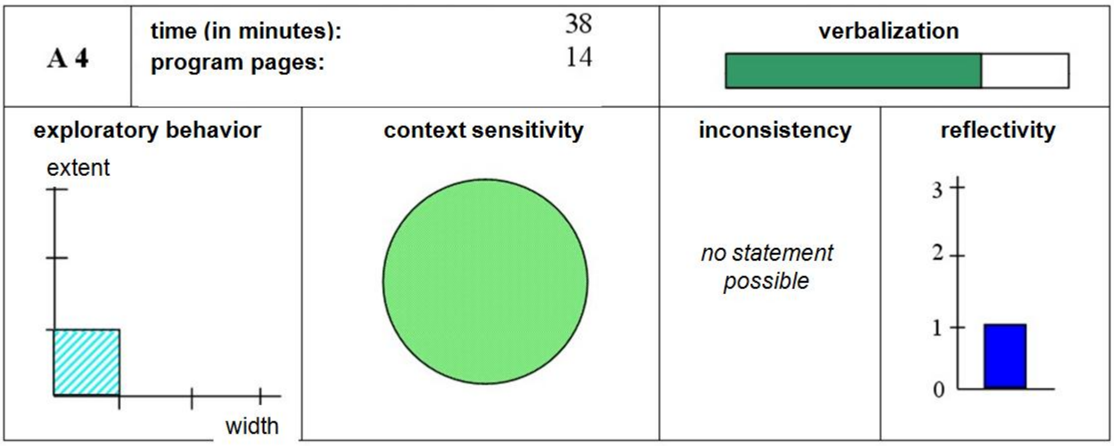


Figure 5.8 *Low SFC* on nearly all components

Additional remarks to the results of the empirical study:

- Differentiated information can be obtained by these different components.
- The components seem to be more or less independent.
- Generally there seems to be a low degree of SFC of nearly all subjects of this study.
- In the experimental group were no specific sensitivity types.
- The degree of sensitivity on all components seems to be related to the amount of "content-sensitivity".

5.4 Discussion and further studies

- A broader range of empirical data with the folding problem as well as more and more different subjects would be useful.
- The validity of the instrument should be studied in more detail.
- There should be used a broader range of research methods.
- It should be analyzed to what extent the behavior of the subjects and the SFC is related to specific pictures of mathematics and mathematics instruction.
- It should be explored to what extend experience with the computer scenario might contribute to rise the SFC of POMI, so to what extent it might help to improve the ability of student teachers to teach mathematical problem-solving.

5.5 Reference to our theoretical framework

This framework has relations as well to the learning level of the pupils as to the learning level of the (beginning) teachers.

- Abilities to find, argue, order, calculate and construct are to be fostered in the pupils by the folding problem. Elements of evaluating and playing are involved too. Of course, the student teachers should have experienced themselves all these activities, too, in order to understand better the statements of the pupils and to react adequately.
- By walking through the network of possible states of the program ("procedures") for several times the student teacher might come to a better understanding of the problem-solving processes of pupils and its variety.

6. Concluding remarks

We presented three studies on mathematical problem solving, which should demonstrate to what considerable extent modern technology might help to improve mathematics education and instruction.

In any case the use of technology especially in mathematics instruction cannot be better than the quality of teachers, who have to fulfill a high standard of competence in education, mathematics and the use of IT. We have to reinforce our efforts in this direction. But on this difficult way IT might help in detecting quality of problem solving (cf. section 4.) and quality of teaching problem solving (section 5.). Both programs might be also used as an *additional* (!) mean for improving the respective competencies.

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